graphical procedure allows exact analysis of a digital control system

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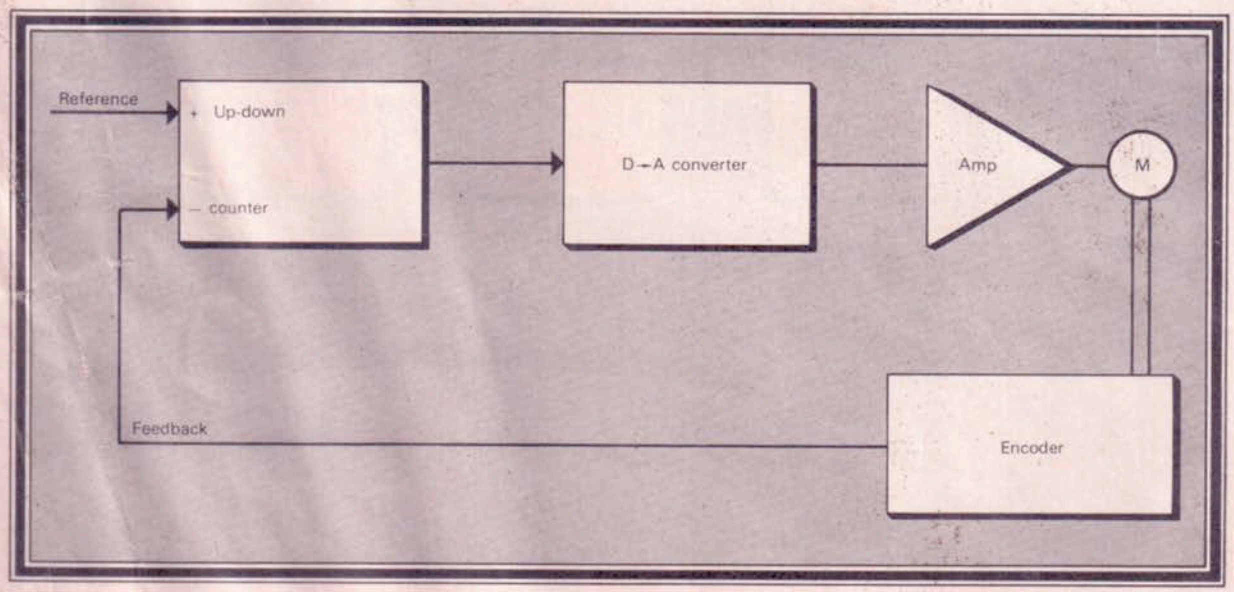
Exact analysis of a digital speed-control system is possible by means of a step-by-step procedure. A mathematical analysis with the aid of the z-transform is not feasible, as the pulses from the encoder measuring the axis position are unevenly spaced (except where the input frequency is constant and the system in steady state); ordinarily, linearisation and the Laplace transform are resorted to. More exact results are obtainable by means of a correction factor. The proposed method provides conditions of non-instability.

A digital speed-control loop is shown schematically in Fig. 1. The frequency of the feedback pulses tends to equality with that of the reference, with the up-down counter serving as equaliser; the pulse-number difference between the two channels is the position error, fed to the motor through an amplifier. The measuring element is a rotating encoder, generating K_E pulses per revolution.

In similar analogue systems, the reference is taken as a voltage, but this is impracticable in the present case, as the feedback would also have to have voltage dimensions, and it would be impossible to formulate the transfer function of the encoder. Accordingly, the reference is taken as a frequency f_R, and the feedback is then also a frequency, f_E. For motor speed ω (in revolutions per second) the transfer function of the encoder is:

Fig. 1 Digital speed-control loop



$$\frac{f_E}{\omega} = K_E \left[\frac{\text{Pulse}}{r}\right] \quad \text{.............................. (1)}$$

It should be borne in mind that the process involves linearisation, as the encoder yields discrete numbers, whereas here its output is apparently continuous.

The counter receives frequencies and yields a number (n). Recalling that

$$n = \int_0^t f_R \mathrm{d}t - \int_0^t f_E \mathrm{d}t \quad \text{.......................... (2)}$$

the transfer function of the counter is $1/s$.

The transfer function of the d.–a. converter is K_c (v/pulse), and that of the amplifier and the motor is:

$$G(s) = \frac{\omega}{E}(s) = \frac{K_M}{1 + s\tau} \quad \text{........................ (3)}$$

where E is the converter output voltage.

The above data yield the overall transfer function for the system, in this case a system of second order. A digital system linearised as above will be referred to as an '*adjoint system*'. (a.s.)

Introduction of correction factor

Fig. 2 compares the true encoder output (a) with the approximation (b), obtained on the basis of the above linearisation. The true output (n_{fa}) is obtained by subtracting a value $\delta(t)$ from the approximate output (n_{fb}), as shown in Fig. 3:

$$n_{fa} = n_{fb} - \delta(t) \quad \text{.............................. (4)}$$

The true $\delta(t)$ is shown in Fig. 4 as a solid line. The increase may be assumed linear with satisfactory accuracy, i.e., $\delta(t)$ is a triangular wave with variable base T_1. As T_1 is unknown, it is proposed to determine $\delta(t)$ as a periodic triangular wave with the same constant period T as the input frequency. (In that case, T_1 is initially larger on starting at constant input frequency, and tends to T at steady state). At steady state $\delta(t)$ is the exact difference to be subtracted, whereas at transient state it is a correction factor yielding more exact results compared with the preceding method. It is worth noting that the mean-square error of n triangular waves with base T equals that of a single wave with base nT, which confirms the satisfactory accuracy of the proposed correction factor. In view of this, and of the fact that the approximate $\delta(t)$ (the correction factor) exceeds its true counterpart, the true result by the digital system (d.s.) lies between that obtained by the a.s. and the result obtained in this section (much closer to the latter).

The Laplace transform of the correction factor is:

$$\delta(s) = \frac{1}{K_E}\left[\frac{1}{Ts^2} - \frac{1/s\, e^{-sT}}{1 - e^{-sT}}\right] \quad \text{.................. (5)}$$

Denoting the principal loop gain by $G_1(s)$, we obtain:

$$n_F(s) = \frac{G_1(s) K_E}{1 + G_1(s) K_E} n_R(s) - \frac{K_E}{1 + G_1(s) K_E} \delta(s) \quad (6)$$

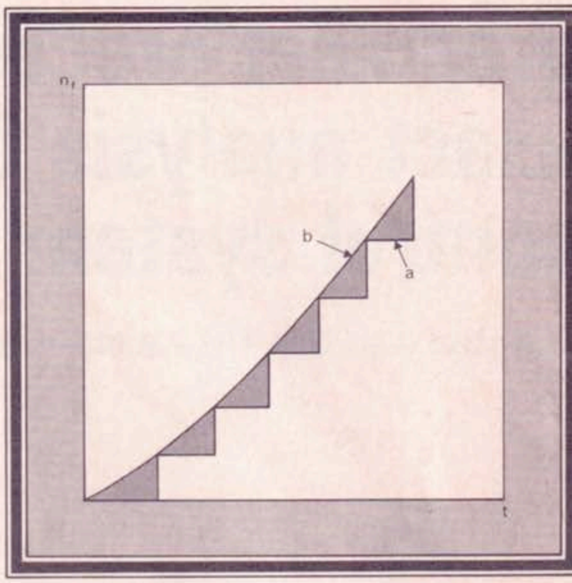
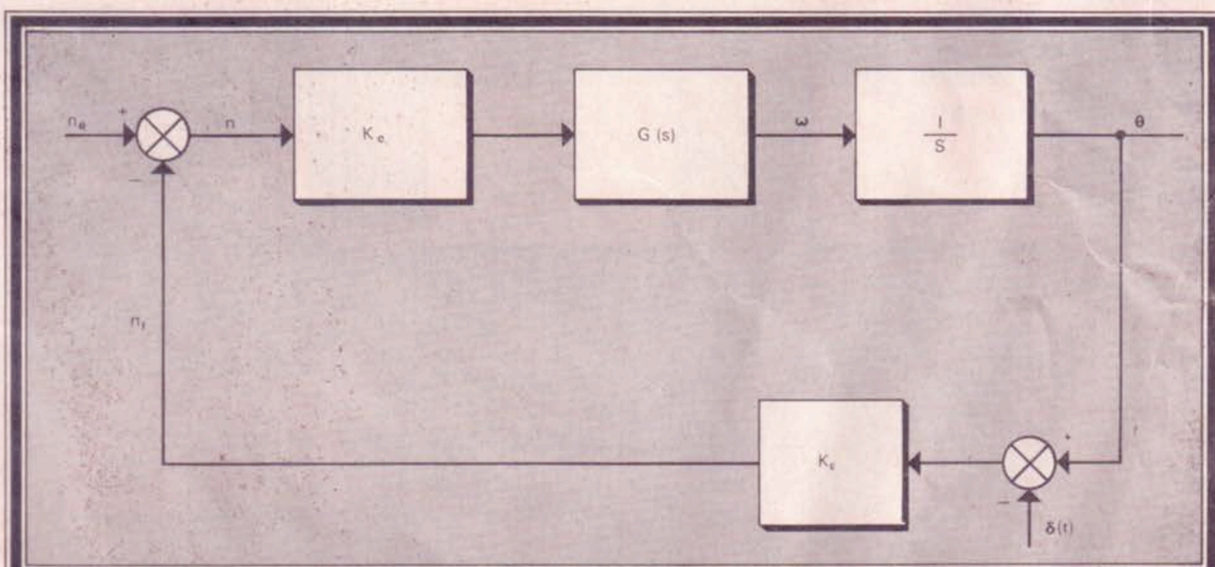


Fig. 2 Comparison of true encoder output (a) with approximation (b)

Fig. 3 Control loop with $\delta(t)$ correction



$$n(s) = \frac{1}{1 + G_1(s) K_E} n_R(s) + \frac{K_E}{1 + G_1(s) K_E} \delta(s) \quad (7)$$

n being the number of required positions on the counter, according to which its number of stages is determined.

Comparison of methods

The methods are compared for constant f_R with period T. By the a.s. system:

$$n(s) = \frac{1}{1 + G_1(s) K_E} \cdot \frac{1}{Ts^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

According to (7):

$$n(s) = \frac{1}{1 + G_1(s) K_E} \cdot \frac{1/s}{1 - e^{-sT}} + \frac{K_E}{1 + G_1(s) K_E} \cdot \frac{1}{K_E}\left(\frac{1}{Ts^2} - \frac{1/s\, e^{-sT}}{1 - e^{-sT}}\right) = \frac{1}{1 + G_1(s) K_E}\left(\frac{1}{Ts^2} + \frac{1}{s}\right) \quad (9)$$

For $G_1(s) = K_1/s$, we obtain, respectively:

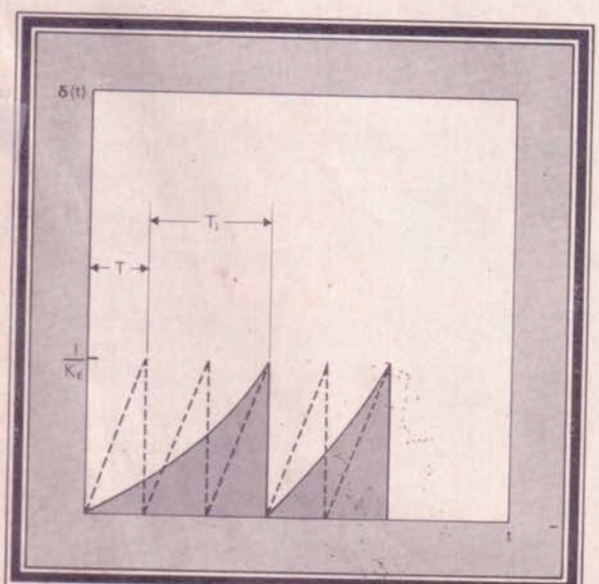
$$n(t) = \frac{f_R}{K}(1 - e^{-Kt}) \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

where $K = K_1 K_E$, and

$$n(t) = \frac{f_R}{K}(1 - e^{-Kt}) + e^{-Kt} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

The table below lists $n(t)$ values by three methods, for $K = 100$ and $f_R = 400$, at intervals of T: in column (a) according to Eq. (10), in column (b) according to Eq. (11), and in column (c) according to the exact stes-by-step method. Obviously, n increases by one digit from point to point, and the data in column (c) were obtained as means of $(n - 1)\,T$ and $(n + 1)\,T$, except for the first pulse. Column (d) lists the exact counter readings at t^+ $(t = n_R T)$, and it may be seen that it is obtainable by rounding off the values in column (b) upwards to the nearest integer.

Fig. 4 True and approximate errors



The table confirms the relative closeness of the results as described above.

Table 1 Comparison of methods

n_R	a	b	c	d
1	0	1	1	1
2	0·88	1·66	1·50	2
3	1·57	2·18	2·12	3
4	2·12	2·59	2·50	3
5	2·53	2·90	2·88	3
6	2·85	3·14	3·02	4
7	3·11	3·33	3·22	4

Introduction of correction factor for general input

The previous sections deal with a step-function input. The general form of the input function is:

$$n_R(s) = \frac{1}{s}\left[1 + \sum_{i=1}^{m} e^{-sT_i}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

This is a Laplace transform for a sequence of frequency-modulated pulses with T_i given by the sample function $n(t)$. By the same considerations as before, the correction factor is

$$\delta(s) = \frac{1}{K_E}\left[n_R(s) - \frac{1}{s}\sum_{i=1}^{m} e^{-sT_i}\right] \quad \ldots\ldots\ldots\ldots (13)$$

and substitution of (12) and (13) into (7) yields

$$n(s) = \frac{1}{1 + G_1(s) K_E}\left[n(s) + \frac{1}{s}\right] \quad \ldots\ldots\ldots\ldots (14)$$

The system output $\theta(s)$ is

$$\theta(s) = \frac{G_1(s)}{1 + G_1(s) K_E}\left[n(s) + \frac{1}{s}\right] \quad \ldots\ldots\ldots\ldots (15)$$

and substitution of $\theta(s) = \frac{\omega(s)}{s}$ and $n_R(s) = \frac{f_R(s)}{s}$ yields

$$\omega(s) = \frac{G_1(s)}{1 + G_1(s) K_E}\left[f_R(s) + 1\right] \quad \ldots\ldots\ldots\ldots (16)$$

In other words, the maximum deviation from the true result if the a.s. is used, is:

$$\Delta\omega(s) = \frac{G_1(s)}{1 + G_1(s) K_E} \quad \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

which is the response of the system to a unit impulse.

Conclusions

Since addition of the response to the impulse in the system output does not affect the nature of the latter, it can be concluded from the preceding section that the d.s. is not unstable if the a.s. is stable. There are, however, cases in which the d.s. may undergo a limit cycle.

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